

Exercise Sheet 6

Exercise 1. Let $P \in \mathbb{C}[X, Y, Z]$ be a non-zero irreducible, homogeneous polynomial. Let $C_P \subset \mathbb{P}_{\mathbb{C}}^2$ be the associated non-singular projective curve. Use the hints to show that C_P is connected for the Euclidean topology.

Hint 1: The Riemann-Roch theorem guarantees for any $x_0 \in C_P$ the existence of a meromorphic function on C_P with a pole only at x_0 of order $\leq r$ for r large enough, and holomorphic elsewhere.

Hint 2: A nonzero meromorphic function on an irreducible algebraic curve cannot vanish on a nonempty Euclidean open set. (*) If you know algebraic geometry, why?

Remark. The statement of Exercise 1 also holds if C_P has singular points.

Exercise 2. Let X, Y be compact connected Riemann surfaces and let $f : X \rightarrow Y$ be a nonconstant holomorphic map. Show that $g(X) \geq g(Y)$.

Exercise 3. (for credit, due on 2 November)

Let X be a compact connected Riemann surface of genus g and let $f : X \rightarrow \mathbb{P}^1$ be a nonconstant holomorphic map. Let r be the number of branch values of f . For $x \in X$ we denote by e_x the ramification multiplicity of f at x .

- (1) (2 points) Let d be the degree of f . Show that $\sum_{x \in X} (e_x - 1) \leq r(d - 1)$.
- (2) (1.5 points) If $r \leq 1$, show that f is a biholomorphism.
- (3) (1.5 points) If $r = 2$, show that $g = 0$.
- (4) Explore the case $r = 3$ (Belyi functions, dessin d'enfant).

Exercise 4. Let X be a compact connected Riemann surface, G a finite group acting holomorphically and properly on X , and $\pi : X \rightarrow Y := X/G$ the quotient map. Suppose there are exactly r orbits of points with non-trivial stabilizer; choose representatives p_1, \dots, p_r with stabilizer orders $m_j > 1$. Prove the formula

$$2g(X) - 2 = |G|(2g(Y) - 2) + |G| \sum_{j=1}^r \left(1 - \frac{1}{m_j}\right).$$

Exercise 5. Let G be a finite group acting by automorphisms on \mathbb{P}^1 and let $\pi : \mathbb{P}^1 \rightarrow Y := \mathbb{P}^1/G$ be the quotient map. Show that π cannot be ramified over exactly one point of Y .

Exercise 6.

Fix an integer $m \geq 2$. Let

$$C = \{[X : Y : Z] \in \mathbb{P}^2 : X^m + Y^m = Z^m\}$$

be the projective Fermat curve of degree m . Compute the genus of C .

Hint: Let $\mu_m = \{\zeta \in \mathbb{C}^\times : \zeta^m = 1\}$ act on C by

$$\zeta \cdot [X : Y : Z] = [X : \zeta Y : Z].$$

Show that C/μ_m is biholomorphic to \mathbb{P}^1 and apply the Riemann-Hurwitz formula from Exercise 4.